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Orderly ε-Homotopies of Discrete Chains

Alexander Thomas Happ
ahapp@utk.edu

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ALEX HAPP

ABSTRACT. I inspect a method of discrete homotopy devised by my advisor, Dr. Conrad Plaut, and Dr. Valera Berestovskii of Omsk State University for constructing covering spaces of arbitrary metric spaces. Itself a simple analog to the notion of path homotopy in the field of Topology, ε-homotopy uses chains governed by some distance ε instead of paths and defines a finite process of steps to imitate the concept of continuous deformation. I prove that when there exists an ε-homotopy between two chains, there must also exist at least one such homotopy with the steps ordered in a specific way. Studying general metric spaces with these discrete chains and homotopies may prove useful, since many geodesic spaces do not have simply connected covering spaces.

1. BACKGROUND

The educated reader may have some familiarity with the study of path homotopy within the field of Topology. The concept is that of a continuous deformation of a path, which can detect gaps in a space. A useful tool in itself, the idea can be extended to detect further the size of these gaps in a discrete and measurable fashion. The context in which we explore this notion is among metric spaces.

Definition 1. *A* metric space *consists of a pair (X,d), with X a set and $d : X \times X \to \mathbb{R}$ a binary function, denoted the metric or distance function, such that the following hold $\forall\, x, y, z \in X$.*

(1) *Symmetry: $d(x, y) = d(y, x)$*
(2) *Positive Definiteness): $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$*
(3) *Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$*

In other words, a metric space is a set of points endowed with the sort of distance one would expect between them. Indeed, for $x, y \in \mathbb{R}$, $d(x, y) = |x - y|$, and in the real plane, we have $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, the familiar distance formula. Utilizing this notion of distance, we make the following construction:

Definition 2. *An* ε-chain *is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ such that $d(x_i, x_{i+1}) < \varepsilon$ for all i.*

One can see how this finite sequence would resemble a chain. A handful of definitions stem from the above: For α an ε-chain with points $\{x_0, x_1, \ldots, x_n\}$, we define the *length* of α to be $L(\alpha) := \sum_{i=1}^{n} d(x_i, x_{i-1})$ with the *size* of α $\nu(\alpha) := n$, and the *reversal* of α is the chain $\overline{\alpha} := \{x_n, ..., x_0\}$. Further, the concept of concatenation with curves has a familiar analog: For any two ε-chains $\alpha = \{x_0, \ldots, x_n\}$ and $\beta = \{y_0, \ldots, y_m\}$ with $x_n = y_0$, we define the *concatenation* $\alpha * \beta := \{x_0, \ldots, x_n = y_0, \ldots, y_m\}$.

Now we may discuss the mechanics of the discrete homotopy.

Definition 3. *A* basic move *on an ε-chain α consists of either adding or removing a single point, provided the resulting chain remains an ε-chain with the same endpoints. A finite sequence of such moves is called an ε-*homotopy*.*

An addition of the point y to a chain is represented with $\{x_0, \ldots, x_{i-1}, \overbrace{\;y\;}, x_i, \ldots, x_n\}$, and a removal of the point y from a chain is represented with $\{x_0, \ldots, x_{i-1}, \underbrace{\;y\;}, x_i, \ldots, x_n\}$. As suggested, a finite sequence of legal additions and removals gives an ε-homotopy, which would consist of a sequence of intermediate ε-chains. Thus, an ε-homotopy η between two ε-chains α and β can be expressed as

$$\eta = \langle \alpha = \eta_0, \eta_1, \ldots, \eta_{m-1}, \eta_m = \beta \rangle$$

with each η_i an ε-chain. We define the *size* of η to be $\nu(\eta) = m$ as with the ε-chains themselves. Furthermore, we call η_i an *addition* if it is obtained from the addition of a point to η_{i-1}, and likewise a *removal* if it is obtained via the removal of a point from η_{i-1}.

If there exists an ε-homotopy from α to β ε-chains, then we say α is ε-homotopic to β, or it is simply homotopic to β. In fact, it can be shown that this homotopy forms an equivalence relation on chains. In this case, we write $\alpha \underset{\varepsilon}{\sim} \beta$ with $[\alpha]_\varepsilon$ denoting the equivalence class of all chains ε-homotopic to α in the space.

Given a metric space X, we can now define $(X_\varepsilon, *)$, the space of ε-homotopy classes of ε-chains with initial point $*$ in X. Its metric is given by $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = |[\overline{\alpha} * \beta]_\varepsilon|$ for $[\alpha]_\varepsilon, [\beta]_\varepsilon \in X_\varepsilon$ and the norm defined by $|[\alpha]_\epsilon| = inf\{L(\gamma) : \gamma \in [\alpha]_\epsilon\}$. The endpoint mapping is denoted by $\phi_\varepsilon : X_\varepsilon \to X$ such that for $\alpha = \{*, x_1, \ldots, x_n\}$, then $\phi_\varepsilon([\alpha]_\varepsilon) = x_n$.

An ε-loop is an ε-chain with identical endpoints, and when such a loop is homotopic to the trivial chain consisting of a single point, we say the loop is null-homotopic. This, of course, necessitates the exception of allowing the removal of one endpoint as the last step of a null-homotopy. At this point, we define for a metric space X $\pi_\varepsilon(X, *)$, the space of homotopy classes of ε-loops with basepoint $*$ in X. It turns out this $\pi_\varepsilon(X, *)$ forms a group under concatenation and carries strong similarities to π_1, the first path-homotopy class from Topology.

2. The Result

Our definition for an ε-homotopy is by no means unique: If there exists an ε-homotopy η between ε-chains α and β, there are likely numerous other ε-homotopies that are equally as legitimate. Included in these is guaranteed a homotopy of a certain form:

Theorem 1. *For all η an ε-homotopy from α to β ε-chains, there exists $\eta' = \langle \alpha = \eta'_0, ..., \eta'_{k-1}, \eta'_k, ..., \eta'_n = \beta\rangle$ an orderly ε-homotopy such that all $i < k$ are additions and all $j \geq k$ are removals. Further, $\nu(\eta) \leq \nu(\eta') \leq 5\nu(\eta) - 4$, or $\nu(\eta) \leq \nu(\eta') \leq \nu(\eta) + 4\nu_a(\eta)$ where $\nu_a(\eta)$ is the number of additions in η.*

Proof. Let $\eta = \langle \alpha = \eta_0, \eta_1, \ldots, \eta_{n-1}, \eta_n = \beta\rangle$ be an ε-homotopy from α to β ε-chains. We can assume WOLOG that the first step is an addition. If there are no removals in η, then we are finished. Otherwise, there is some η_k the first removal. If all $i > k$ are also removals, then we're finished. If not, there exists $l > k$ an addition of the point z between x and y:

$$\eta_l = \{x_0, ..., x_p, x, \overbrace{z}, y, x_{p+1}, ..., x_m\}$$

To maintain generality, we include in our consideration points w_i that will be removed between η_k and η_l. We start with η_{k-1}:

$$\eta_{k-1} = \{x_0, ..., x_p, x, w_1, ..., w_q, y, x_{p+1}, ..., x_m\}$$

Now we perform the following additions immediately following our η_{k-1}:

$$\begin{aligned} \eta'_{k_1} &= \{x_0, ..., x_p, x, w_1, ..., w_q, y, \overbrace{y}, x_{p+1}, ..., x_m\} \\ \eta'_{k_2} &= \{x_0, ..., x_p, x, w_1, ..., w_q, y, \overbrace{x}, y, x_{p+1}, ..., x_m\} \\ \eta'_{k_3} &= \{x_0, ..., x_p, x, w_1, ..., w_q, y, x, \overbrace{z}, y, x_{p+1}, ..., x_m\} \end{aligned}$$

We have effectively inserted a point z between x and y using only duplications of points already involved in the homotopy. Now we proceed with the removals η_k through η_l, which will leave our new additions of x, y, and z untouched. Then we perform the following removals, starting with the new η_l containing our recent additions:

$$\begin{aligned} \eta'_l &= \{x_0, ..., x_p, x, \underbrace{y}, x, z, y, x_{p+1}, ..., x_m\} \\ \eta'_{l_1} &= \{x_0, ..., x_p, \underbrace{x}, x, z, y, x_{p+1}, ..., x_m\} \\ \eta'_{l_2} &= \{x_0, ..., x_p, x, z, y, x_{p+1}, ..., x_m\} \end{aligned}$$

This final chain, η'_{l_2}, is equivalent to the η_l in the original homotopy above. Now if all the following steps are removals, we are finished. If not, we detect this $t > l$ a removal and perform the same process. As the homotopy by definition contains a finite number of points, the repetitions will cease, yielding some η' such that all additions are applied prior to any removals.

As this process involves only adding more steps to the homotopy, we clearly have $\nu(\eta) \leq \nu(\eta')$. To obtain a bound on the lengthening of this process, observe that each iteration of the algorithm described adds 4 steps to the original homotopy. Further, notice that an iteration is triggered only when an addition occurs subsequent to a removal. Thus, the process of creating η' adds at most $4\nu_a(\eta)$ steps to η, yielding $\nu(\eta') \leq \nu(\eta) + 4\nu_a(\eta)$, where $\nu_a(\eta)$ is the number of additions in η. The greatest number of additions possible subsequent to a removal is exactly $\nu(\eta) - 1$ (if η consists of one removal at the start and all additions following), so we get the rougher bound in $\nu(\eta') \leq \nu(\eta) + 4(\nu(\eta) - 1) = 5\nu(\eta) - 4$.

□

This result (reached with the guidance of my advisor, Dr. Conrad Plaut) has a number of possible applications where actual ε-homotopies are involved. For example, we know that the following diagram is commutative for all $\varepsilon > 0$ via path homotopy and covering space theory:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{f}(t)"] \arrow[dr, "f(t)"'] & (S^1)_\varepsilon \arrow[d, "\phi_\varepsilon"] \\ & S^1 \end{tikzcd}$$

where f is the usual mapping onto the circle by $f(t) = cos 2\pi t + sin 2\pi t$ and ϕ_ϵ is the endpoint mapping. But in the context of discrete homotopy, a large part of the proof consists in proving that $\pi_\varepsilon(S^1)$ is isomorphic to $\mathbb{Z}$ for all $\varepsilon > 0$, and an elementary proof of injectivity of the mapping requires the existence of an orderly homotopy.

Of course, the method of discrete homotopy finds its richest application among metric spaces which do not necessarily have nice properties such as local path connectedness. This result will surely find utility there.

References

[1] **Dr. Conrad Plaut**, *UTK Summer REU Notes*, Summer 2010.